B = ln (A_0/A_L)
C = concentration as a function of position Z and time t, g./ml.
$C_{(L,\,t)}$ = concentration at $Z = L$ and $t = t$, g./ml.
C_M = mean concentration, g./ml.
$C_{(0,\,t)}$ = concentration at $Z = 0$, and $t = t$, g./ml.
$C_{(\infty,\,t)}$ = concentration at $Z = \infty$ and $t = t$, g./ml.
D_L = axial diffusion coefficient, sq. cm./sec.
d_p = diameter of particle, cm.
L = packed bed length, cm.
n = number 1, 2, 3, . . .
N_{Pe} = Peclet number $= d_p U/D_L$
N_{Re} = Reynolds number $= d_p U \rho/\mu$
N'_{Re} = Reynolds number $= d_p U' \rho/\mu$
t = time, sec.
U = interstitial velocity in bed, cm./sec.
U' = velocity based on empty tube, cm./sec.
Z = longitudinal distance in bed, cm.

Greek Letters

θ = distance between intersection of 50% points, cm.
τ = wave period, sec.
ξ = phase angle, radians
μ = viscosity, poise
ρ = density, g./ml.
ω = angular frequency, radians/sec.

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The Role of Porosity in Filtration:

IV. Constant Pressure Filtration

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Certain assumptions which have previously served as a basis for the conventional equations employed in constant pressure filtration are shown to be in error. It is demonstrated that the specific filtration resistance, the ratio of the mass of wet to mass of dry cake, and the rate of flow, $q = dv/d\theta$, are not constant as has been assumed. In an example it is shown that q undergoes an eightfold variation as the liquid flows from the cake surface through to the medium.

Since the product αq appears in the basic differential equation, incorrect values of q lead to errors in the calculated values of α arising from experimental data. The errors are significant when thick slurries are employed.

New partial differential equations are presented for flow through compressible media in which q varies with cake thickness. Modifications of the conventional constant pressure equations are presented.

In the conventional analysis of constant pressure cake filtration (*1, 3, 4*), three approximations are made which may be invalid under certain circumstances. In the Ruth filtration equation

$$q = \frac{dv}{d\theta} = \frac{1}{\mu}\,\frac{g_c\,p}{\left[\dfrac{\alpha s \rho}{1-ms}\,v + R_m\right]} \qquad (1)$$

It is generally assumed for constant pressure operation that the filtration resistance is constant, the ratio of the mass of wet to mass of dry cake is constant, and at any instant the rate of flow $dv/d\theta$ is constant throughout the cake. Analysis of basic phenomena indicates that these assumptions must be modified and that discrimination must be employed for correct use of Equation (1). The first two postulates are primarily in error in the initial period of filtration. Consequently it is in rotary filtration and in abbreviated experimental determinations of cake characteristics where the time of filtration is frequently short that the employment of constant values of α and m may lead to erroneous results. The assumption involving the constancy of $dv/d\theta$ may be strikingly in error for thick slurries.

PRESSURE AT MEDIUM

In Figure 1 a filter cake is illustrated in which flow of fluid takes place from left to right, and distance is measured from the surface of the cake. As the liquid flows frictionally through the compressible, porous media, p_s drops until it reaches the value p_1 at the interface of the cake and supporting medium.

The pressure at the medium is defined by

$$g_c\,p_1 = \mu\,R_m\,q_1 = \mu\,R_m\,\frac{dv}{d\theta} \qquad (2)$$

where $q_1 = dv/d\theta$ is the rate of filtrate flow in (cubic feet)/(square feet)(second). The relationship of time to pressure p_1 at the medium and the pressure drop across the cake $(p - p_1)$ is illustrated in Figure 2 for talc filtered at a constant pressure of 5 lb./sq. in. Initially when there is no cake, the entire pressure drop is across the medium and $p = p_1$. As $dv/d\theta$ decreases with time, p_1 falls in accord with Equation (2) and the pressure drop $(p - p_1)$ across the cake builds up.

In general a medium should be chosen to give a minimum resistance consistent with the production of satisfactory clarity. Grace (*2*) has indicated that a filter medium exhibiting a resistance equivalent to no more than 0.01 in. of cake can usually be selected with the result that the pressure drop across the medium becomes a negligible portion of the total pressure drop for a major portion of the filter cycle.

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In Figure 2 the curve illustrating the pressure at the medium rises and falls as the resistance of the medium is changed, and the cake resistance remains constant. With relatively small values of R_m the pressure at the medium rapidly drops to low values.

The thickness of the cake is given by the mass of dry cake per square foot divided by the dry cake density $\rho_s (1 - \epsilon)$. The ratio R_m/α represents the mass of dry solids per square foot equivalent to the resistance of the medium. Consequently the equivalent thickness of the medium is given by $R_m/\alpha\rho_s(1 - \epsilon_{av})$ or $mR_m/\alpha\rho_{av}$. Since m decreases and ρ_{av} increases with time, the equivalent cake thickness is a maximum at the start of filtration; however it rapidly decreases to an approximately constant value. For the example illustrated in Figure 2 the initial equivalent medium thickness is 0.41 in., while the ultimate value approached is 0.05 in. The value of $R_m = 1.0\ (10^{10})$ was chosen so that it would be approximately one-tenth of α for talc at 4 lb./sq. in. For the example of Figure 3 the medium resistance was selected as 2.0 (10^{10}), and the ratio R_m/α approached a value of 0.07 as the pressure increased to 40 lb./sq. in. Initially the equivalent thickness of the medium was 0.82 in., while a value of 0.027 in. was finally reached.

FILTRATION RESISTANCE

In constant pressure filtration (3) the average value of α used in Equation (1) has frequently been used in the form

$$\alpha = \frac{p}{\displaystyle\int_0^p \frac{dp_s}{\alpha_x}} \qquad (3)$$

the relation between α_x and p_s may be obtained in a compression-permeability cell (*1, 3*) or by indirect calculation (*8*) based upon actual filtration data. Equation (3) is valid when the medium resistance is negligible or when the rate has dropped to a point where p_1 is small. A more exact expression for an average α is given by (*1, 6*)

$$\alpha = \frac{p - p_1}{\displaystyle\int_0^{p-p_1} \frac{dp_s}{\alpha_x}} = \frac{p - p_1}{\displaystyle\int_0^{p-\mu R_m q} \frac{dp_s}{\alpha_x}} \qquad (4)$$

where $(p - p_1)$ represents the pressure drop across the cake. Whereas α is a function of p alone in Equation (3), it is a function of p and $q = dv/d\theta$ or the time in Equation (4) in addition to the parameters μ and R_m (*6*).

Equation (3) yields a value of α which would remain unchanged during the course of a constant pressure filtration. In constant pressure filtration literature, particularly in earlier years, the assumption was made that the average value of α was constant and that the volume vs. time curve was a perfect parabola (*4*). Under a wide variety of conditions the parabolic relationship is approximately valid. However if the more correct average value of α as defined by Equation (4) is employed, it can be seen that the specific resistance will vary throughout a constant pressure filtration. Under certain circumstances the variation in α may be sufficient to cause a noticeable deviation in the parabolic relationship, particularly at small time intervals.

In Figure 3 calculated values of the specific resistance as defined by Equation (4) are plotted against the time for the constant pressure filtration of talc. [Same parameters used as in Example 1, Paper No. III of this series (*8*).] The curve for α starts at a low value, corresponding to zero compressive pressure in the first layer of cake, and rises very rapidly. The rapidity of approach to the limiting value which is equivalent to that given by Equation (3) depends upon the slurry concentration and other parameters. In constant pressure literature a great proportion of reported experimental data has been restricted to filtrations involving only a few minutes duration. It appears that calculation based on such data would have to be treated with caution. Other complications in checking theory with experiment arise in that at the start of a constant pressure filtration there may be experimental difficulties, such as adjusting pressure, and in addition cake filtration may be proceeded by blocking and bridging action without immediate cake formation. The combined theoretical and experimental problems involved at the beginning of a constant pressure filtration point to the need for lengthy runs if complete confidence is to be placed in calculated results.

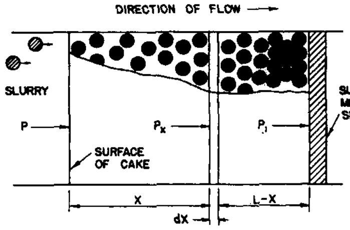

Fig. 1. Schematic diagram of filter bed.

AVERAGE CAKE POROSITY

Porosity varies throughout the cake as a function of time and thickness as illustrated in Figure 4, where ϵ is plotted against x for constant value of θ. Data used for the graph represent calculated values and are purely illustrative. The first infinitesimal layer at the cake surface has a porosity and specific resistance corresponding to zero compressive pressure. At each instant of time the porosity drops throughout the cake until the medium is reached where it has its least value. The cake thickness increases, and at a given distance from the cake surface the porosity increases as time goes on. As the pressure drop across the cake increases, the porosity at the medium decreases and eventually reaches a minimum value equal to a porosity determined by the maximum applied pressure.

It is instructive to replot the data showing the porosity as a function of the distance $(L - x)$ from the medium as indicated in Figure 5. The first layer of solids deposited on the medium has a porosity corresponding to point *A*. As time proceeds, the porosity at the medium decreases and approaches a minimum value at *B* which is determined by the total filtration pressure. The average value of the porosity represented by the dotted line approaches a limiting value as the cake becomes indefinitely thick. A replot of the data showing ϵ as a function of x/L is presented in Figure 6.

The ratio of total mass of wet cake to total mass of dry cake is related to the porosity by

$$m = 1 + \frac{\rho\epsilon_{av}}{\rho_s\ (1 - \epsilon_{av})} \qquad (5)$$

Obviously as ϵ_{av} decreases, m will also decrease. In Figure 7 a plot of calculated values of m vs. θ for the filtration of several materials is shown. The value of m changes with time, and the basic assumption that the factor (1—ms) in Equation (1) can be treated as a constant during mathematical integrations is false. Since m may vary widely at the beginning of filtration, it becomes necessary to re-evaluate the theoretical aspects of short operations. As m enters calculations through the factor $(1 - ms)$, the greatest errors in assuming m constant will arise when s is large. In the example used in Figure 2 s had the small value 0.003, and (1—ms) was nearly unity; consequently it was possible to use Equation (1) on the assumption that $(1 - ms)$ is constant. In rotary filtration (*5*), where thick slurries are normally employed, the assumption of constant m must be approached cautiously. For most filtrations however the value of m will approach its limiting value within a few seconds.

FLOW RATE

In conventional derivations of filtration formulas the rate of flow has been considered constant throughout the cake at any instant. Mathematically $\partial q/\partial x$ has been assumed zero and q independent of x. The rate of filtration usually is related to other variables by writing the following material balance on the basis of 1 sq. ft.:

Total mass of slurry = mass of cake + mass of filtrate

$$\frac{w}{s} = mw + \rho v \tag{6}$$

Solving for v one obtains

$$v = \frac{1 - ms}{\rho s} w \tag{7}$$

Differentiating v with respect to time one gets the rate of filtrate flow q_1 at the exit of the cake; thus

$$q_1 = \frac{dv}{d\theta} = \left(\frac{1 - ms}{\rho s}\right)\frac{dw}{d\theta} - \frac{w}{\rho}\frac{dm}{d\theta} \tag{8}$$

In the past it has been customary to consider m constant with v directly proportional to w. Inspection of Figure 7 indicates that $dm/d\theta$ is negative and approaches zero rapidly. Only after $dm/d\theta$ becomes negligible is the rate of filtrate flow proportional to the rate of deposit of solids $dw/d\theta$.

It is instructive to compare the rate of flow q_i in an infinitesimal layer of cake at the surface with the exit rate of filtration q_1 (Figures 8 and 9). The total pounds of slurry per square foot per second approaching the surface of the cake is given by $(1/s)\,dw/d\theta$, and the rate of liquid flow in the slurry may be presented in the form

$$q_o = \frac{\text{cu. ft. of liquid}}{(\text{sq.ft.})(\text{sec.})} = \frac{1 - s}{\rho s}\frac{dw}{d\theta} \tag{9}$$

At any instant the flow rate varies throughout the cake as indicated in Figure 9 from a minimum value of q_i at the surface to a maximum value q_1 at the exit of the cake. The rate at which liquid is retained in the surface layer is given by

$$\frac{\text{cu. ft. of liquid deposited at surface}}{(\text{sq. ft.})(\text{sec.})} = \frac{m_i - 1}{\rho}\frac{dw}{d\theta} \tag{10}$$

where $dw/d\theta$ may be obtained from Equation (8). Subtracting Equation (10) from (9) one gets the net rate of flow at the cake surface; thus

$$q_i = \left[\frac{1 - s}{\rho s} - \frac{m_i - 1}{\rho}\right]\frac{dw}{d\theta} = \frac{1 - m_i s}{\rho s}\frac{dw}{d\theta} \tag{11}$$

when one substitutes for $dw/d\theta$

$$q_i = \left[\frac{1 - m_i s}{1 - ms}\right] q_1 + \left[\frac{w(1 - m_i s)}{\rho(1 - ms)}\right]\frac{dm}{d\theta} \tag{12}$$

Since $m < m_i$, the multiplier of q_1 is less than unity. In addition the derivative $dm/d\theta$ is negative, and the second term is also negative. Consequently q_1 is always greater than q_i, and the rate of flow increases as the fluid passes through the cake. A physical explanation may be derived from the continued compressive action at any point in the cake. In Figure 5, where the porosity is plotted against the distance from the medium, it can be seen that the porosity decreases with time at any point in the solid. When the porosity or liquid content is decreased, liquid is literally squeezed out of each differential element. Since flow is additive, the flow rate necessarily increases as the liquid approaches the medium.

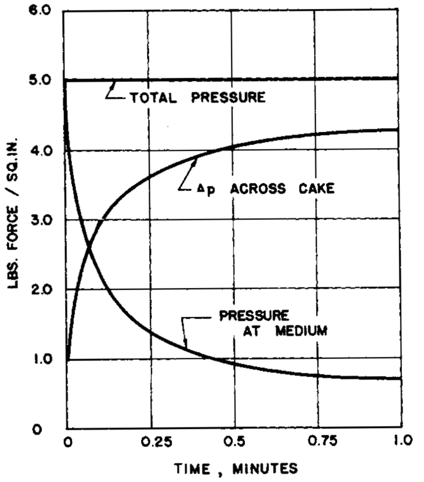

Fig. 2. Variation of pressure drops across cake and medium.

Even after the filter cake has virtually reached its ultimate average moisture content and $dm/d\theta$ is zero, the flow rate continues to vary throughout the cake. While it may appear paradoxical that the flow rate is variable in a cake having constant average moisture content, it must be remembered that the surface layer deposited by the slurry always possesses a high porosity and the cake is continually being compressed while it grows in thickness. If $dm/d\theta$ is zero, Equation (12) becomes

$$q_1 = \frac{1 - ms}{1 - m_i s} q_i \tag{13}$$

If s is sufficiently small, there will be little difference between q_1 and q_i. However in many filtrations the products ms and $m_i s$ may differ significantly. If the value of m from Equation (5) is substituted in (13) there results

$$\frac{q_1}{q_i} = \frac{1 - X f_{av}/\sigma}{1 - X f_i/\sigma} \tag{14}$$

Equation (14) can be employed for estimating the variation in q throughout the cake. Unfortunately porosity vs. pressure data are virtually nonexistent in the region near zero compressive pressure, and extrapolations, which may be inaccurate for highly compressible materials, must be employed. In Table 1 calculations of the ratio q_1/q_i for several materials filtered in an aqueous suspension at 100 lb./sq. in. are presented.

As s increases and the slurry becomes more concentrated, the ratio of q_1/q_i approaches high values indicating that there is a large variation of flow within the cake. The use of the customary filtration equations would have to be seriously questioned on a theoretical basis when q_1/q_i varies considerably. In practice the largest variations would normally occur in rotary filtration where the slurries may be highly concentrated and the time of filtration less than a minute.

Calculations of α based on experimental data have generally been obtained from an integrated form of Equation (16). Since αq occurs as a product in Equation (16) and also in Equation (1), use of incorrect values of q will lead to erroneous values of α. In general exit values of the flow rate have been employed in filtration literature, and, as the exit flow rate is the maximum in the cake, values calculated for the specific resistance have been low. Fortunately many experimental filtrations have been carried out with dilute slurries, and the corrections should not be large. For dilute slurries the maximum possible fractional error is given approximately by

$$\text{error} = \frac{q_1 - q_i}{q_1} = \frac{X(f_i - f_{av})}{\sigma - X f_{av}} \tag{15}$$

In many cases inherent experimental errors have been greater than any corrections which might be applied for small variations in the flow rate. Nevertheless great caution should be applied in utilizing data appearing in the literature, particularly where porosity data are unavailable.

The data of Grace (*1*) for the filtration of zinc sulfide may be used to illustrate a large variation in flow rate

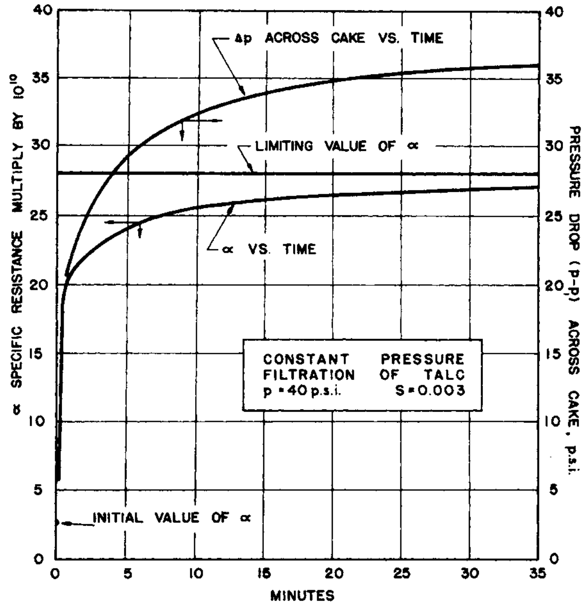

Fig. 3. Filtration resistance vs. time in constant-pressure filtration.

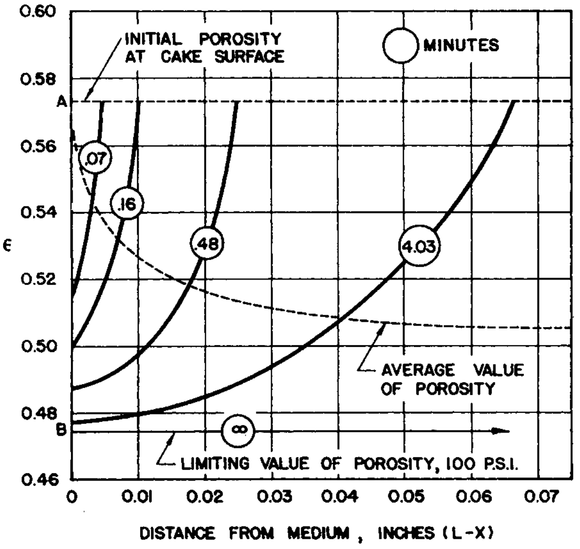

Fig. 5. Porosity vs. distance from medium.

throughout a cake in an experimental constant pressure filtration. The following data are needed:

Filtration pressure	11.46	lb./sq. in.
Average cake porosity	0.864	
Surface porosity at 0 pressure (extrapolated)	0.94	
Pounds of dry solids per cubic foot of filtrate	24.1	
Specific gravity (approximate)	4.09	

With these values the slurry concentrations and the average ratio of wet to dry cake are calculated to be 0.194 and 2.56 respectively. When one uses an extrapolated surface porosity of 0.94 (porosity is 0.9 at lowest experimental pressure of about 1.1 lb./sq.ft.), the value of m_i becomes 4.84. Utilizing Equation (7) one gets

$$q_1 = \frac{1-(2.56)(0.194)}{1-(4.84)(0.194)} q_i = 8.27 q_i$$

which states that an eightfold variation in rate takes place across the cake. No conclusions can be drawn as to how close an average rate within the cake compares with either the exit or entrance rates. The porosity variation* determines the way in which q changes with distance. Since the porosity vs. cake thickness function can take widely different forms with rapid porosity changes either at the surface or at the medium, no generalization can be drawn as to what type of average of q_1 and q_i can be used in the basic filtration equation.

BASIC DIFFERENTIAL EQUATIONS OF FILTRATION

The basic empirical equation describing flow through a porous, compressible solid is generally written as (7)

$$-g_c \frac{dp_x}{dx} = g_c \frac{dp_s}{dx} = \alpha_x \mu \rho_s (1-\epsilon) q \tag{16}$$

or by the elimination of dx by $dw_x = \rho_s(1-\epsilon)dx$

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* Methods for calculating porosity as a function of distance will be presented in the next paper of this series.

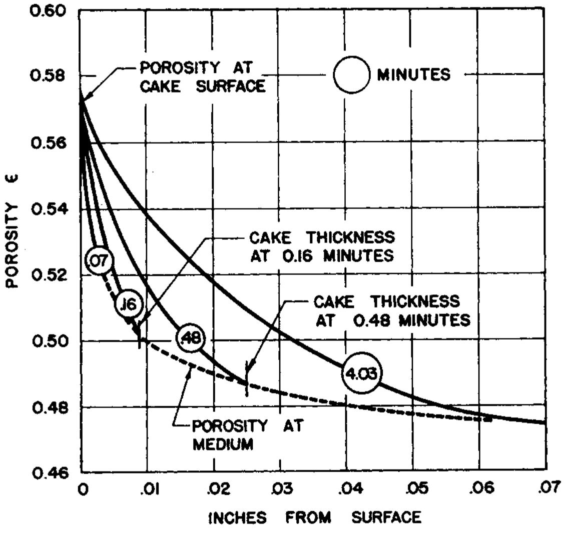

Fig. 4. Porosity vs. distance from surface.

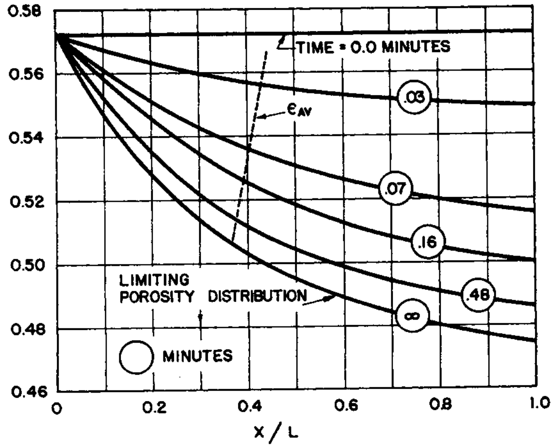

Fig. 6. Porosity vs. x/L.

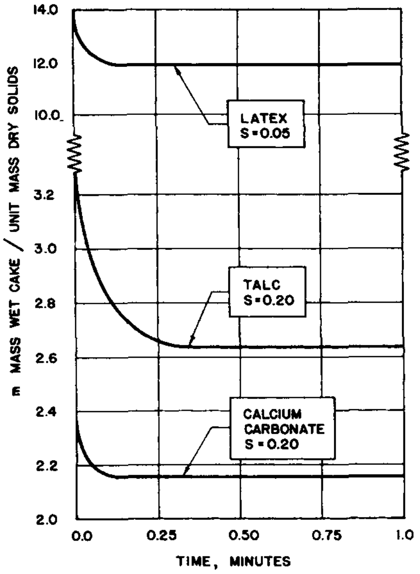

Fig. 7. Variation with time of ratio of mass of wet cake to mass of dry cake.

TABLE 1

Substance	Reference	Specific gravity	ϵ_i (0 lb./sq.in.)	ϵ (100 lb./ ......)	q_1/q_i $s=0.01$	$s=0.05$	$s=0.10$	$s=0.20$
Hyflo	(*6*)	2.3	0.89	0.825	1.01	1.08	1.27	4.07
Kaolin	(*6*)	2.60	0.696	0.477	1.01	1.03	1.07	1.17
Talc	(*1*)	2.68	0.90*	0.67	1.03	1.17	1.46	5.06
Dow	(*1*)	1.00	0.95*	0.45		s=0.02	s=0.03	s=0.04
polystyrene latex					1.21	1.55	2.18	38.0

* Extrapolation to zero pressure. Values sensitive to method of extrapolation and may be considered as approximate.

$$g_c \frac{dp_s}{dw_x} = \alpha_x \mu q \qquad (17)$$

In past investigations (*1, 6, 7, 8*), Equations (16) and (17) have been integrated on the basis of q being constant throughout the cake at any instant; that is q has been assumed to be a function of the time θ but not the distance x. However if the average porosity of the cake is decreasing, q will be varying from the cake surface through the solid reaching its maximum value at the interface between the solid and supporting medium; it becomes necessary to write new differential equations. For a given mass of filter cake, limits on the pressure drop may be obtained by substituting q_i and q_1 for q in (17).

A liquid material balance over a differential section of the cake on a unit area basis yields

$$\frac{\text{Rate out}}{\text{area}} - \frac{\text{Rate in}}{\text{area}} = \text{(rate of change of porosity)} \times \frac{\text{volume}}{\text{area}}$$

$$q\,(\text{out}) - q\,(\text{in}) = \Delta q = \frac{\epsilon\,(\text{initial}) - \epsilon\,(\text{final})}{\Delta\theta} \Delta x \qquad (18a)$$

or, when one transfers Δx to the left-hand side and takes the limit

$$\frac{\partial q}{\partial x} = -\frac{\partial \epsilon}{\partial \theta} \qquad (18b)$$

This equation may also be obtained by writing the hydrodynamical equation of continuity. Since ϵ is a function of p_s alone, Equation (18*b*) may be rewritten as

$$\frac{\partial q}{\partial x} = -\frac{d\epsilon}{dp_s} \frac{\partial p_s}{\partial \theta} \qquad (19)$$

The quantity $\partial\epsilon/\partial\theta$ is negative since the porosity decreases at each point with respect to time; consequently $\partial q/\partial x$ is positive, indicating that q increases with x.

Equations (16) and (19) represent simultaneous equations with q and p_s as dependent variables and x and θ as independent variables. To eliminate q between Equations (16) and (19) the former equation is first differentiated with respect to x to give

$$g_c \frac{\partial^2 p_s}{\partial x^2} = \mu \rho_s \frac{\partial}{\partial x} [\alpha_x (1 - \epsilon) q]$$

$$= \mu \rho_s \frac{d}{dp_s} [\alpha_x (1 - \epsilon)] \frac{\partial p_s}{\partial x} q$$

$$- \mu \rho_s \alpha_x \ (1 - \epsilon) \frac{\partial q}{\partial x} \qquad (20)$$

Since α_x and ϵ are assumed to be functions of p_s alone, it is not necessary to use partial differential notation in differentiating $\alpha_x(1-\epsilon)$. Eliminating q and $\partial q/\partial x$ by using Equations (16) and (19) one gets

$$g_c \frac{\partial^2 p_s}{\partial x^2} = \frac{g_c}{\alpha_x(1-\epsilon)} \left(\frac{\partial p_s}{\partial x}\right)^2 \frac{d}{dp_s}$$

$$[\alpha_x(1-\epsilon)] - \mu\rho_s \alpha_x (1-\epsilon)$$

$$\left(\frac{d\epsilon}{dp_s}\right) \left(\frac{\partial p_s}{\partial \theta}\right) \qquad (21)$$

Equation (21) may be rewritten as

$$g_c \frac{\partial^2 p_s}{\partial x^2} = g_c \left(\frac{\partial p_s}{\partial x}\right)^2 \frac{d}{dp_s} \ln[\alpha_x (1-\epsilon)]$$

$$- \mu \rho_s \alpha_x (1-\epsilon) \left(\frac{d\epsilon}{dp_s}\right) \left(\frac{\partial p_s}{\partial \theta}\right) \qquad (22)$$

The independent variable x may be replaced by w_x with the following substitutions:

$$\frac{\partial p_s}{\partial x} = \rho_s (1-\epsilon) \frac{\partial p_s}{\partial w_x} \qquad (23)$$

When one obtains the second derivative

$$\frac{\partial^2 p_s}{\partial x^2} = \rho_s \frac{\partial p_s}{\partial w_x} \frac{d}{dp_s} (1-\epsilon) \left(\frac{\partial p_s}{\partial x}\right)$$

$$+ \rho_s (1-\epsilon) \frac{\partial^2 p_s}{\partial w_x^{\,2}} \frac{dw_x}{dx} \qquad (24)$$

Substituting for $\partial p_s/\partial x$ and dw_x/dx one gets

$$\frac{\partial^2 p_s}{\partial x^2} = \rho_s^{\,2} (1-\epsilon) \left(\frac{\partial p_s}{\partial w_x}\right)^2 \frac{d(1-\epsilon)}{dp_s}$$

$$+ \rho_s^{\,2} (1-\epsilon)^2 \frac{\partial^2 p_s}{\partial w_x^{\,2}} \qquad (25)$$

Substitution in (21) leads to

$$g_c \frac{\partial^2 p_s}{\partial w_x^{\,2}} = g_c \left(\frac{\partial p_s}{\partial w_x}\right)^2 \frac{d}{dp_s} \ln \alpha_x$$

$$- \frac{\mu \alpha_x}{\rho_s} \frac{d}{dp_s} \ln(1-\epsilon) \frac{\partial p_s}{\partial \theta} \qquad (26)$$

Equation (26) can also be derived by direct use of Equations (17) and (19).

The preceding equations are based upon the assumption that α_x and ϵ are functions of p_s alone, which is equivalent to assuming that equilibrium porosities are reached instantaneously with changes in pressure. Since it is known (*6*) that porosity changes do not occur instantaneously, Equations (22) and (26) can be viewed only as approximations in the march toward obtaining the best differential equations for flow through compressible porous media.

The last term in Equation (22) or (26) arises because of the assumed variation of q with x. If $\partial p_s/\partial\theta$ is placed equal to zero, the remaining terms in either (22) or (26) may be reduced to the conventional filtration equation by reversing the steps given in Equations (17) through (21).

The solution of the nonlinear partial differential Equation (26), in which the coefficients are obtained from experimental data, does not lend itself to simple solution. Numerical techniques must be employed for obtaining p_s as a function of x and θ.

APPROXIMATE METHODS

Upper and lower limits on the variation of p_s in the filter cake can be obtained by assuming that the flow rate remains constant throughout the cake and equals either the exit rate q_1 or the entrance rate q_i. Since q_1 is larger than the rate at any point within the solid, use of q_1 throughout the cake will lead to the largest value of pressure drop.

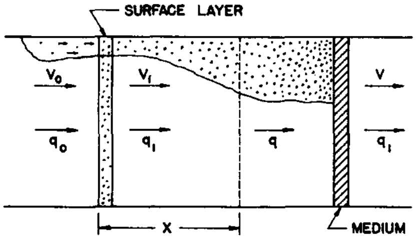

Fig. 8. Flow rates within filter bed.

If the value $q(x,\theta)$ in Equation (17) is replaced by $q_1 = q(L,\theta) = q(w,\theta)$ or by $q_i = q(0,\theta)$, it is possible to integrate the equation treating the rate of flow as a constant. In order to calculate q_1 and q_i it is necessary to know both the porosity at the surface and its average value. A method for calculating pressure drops will be outlined on the assumption that the average porosity changes with time but q is constant throughout the cake. Solving for dw_x in Equation (17) and integrating one gets

$$\int_o^w dw_x = w = \frac{g_c}{\mu q} \int_o^{p-p_1} \frac{dp_s}{\alpha_x} \tag{27}$$

Elimination of w in terms of v by means of Equation (7) gives

$$v = \frac{g_c(1-ms)}{\mu\ \rho\ s\ q} \int_o^{p-p_1} \frac{dp_s}{\alpha_x} \tag{28}$$

A changing value of m will be used in Equation (28). Substitution of q_i and q_1 for q in (28) will yield limiting values of v for a given pressure drop $p - p_1$. In order to calculate m it is necessary to obtain the average porosity. If L is the thickness of the cake, then

$$\epsilon_{av} = \frac{1}{L} \int_o^L \epsilon dx \tag{29}$$

Or changing the variable of integration one obtains

$$\epsilon_{av} = \frac{1}{L} \int_o^{p-p_1} \epsilon \left(\frac{dx}{dp_s}\right) dp_s \tag{30}$$

The term dx/dp_s can be obtained from Equation (16) and substituted in Equation (30) to yield

$$\epsilon_{av} = \frac{g_c}{\mu\ \rho_s\ q} \frac{1}{L} \int_o^{p-p_1} \frac{\epsilon\ dp_s}{\alpha_x(1-\epsilon)} \tag{31}$$

The thickness of the cake can be found by integrating dx from zero to L in Equation (16) as follows:

$$\int_o^L dx = L = \frac{g_c}{\mu\ \rho s\ q} \int_o^{p-p_1} \frac{dp_s}{\alpha_x(1-\epsilon)} \tag{32}$$

When one substitutes for L in Equation (31)

$$\epsilon_{av} = \frac{\displaystyle\int_o^{p-p_1} \frac{\epsilon\ dp_s}{\alpha_x(1-\epsilon)}}{\displaystyle\int_o^{p-p_1} \frac{dp_s}{\alpha_x(1-\epsilon)}} \tag{33}$$

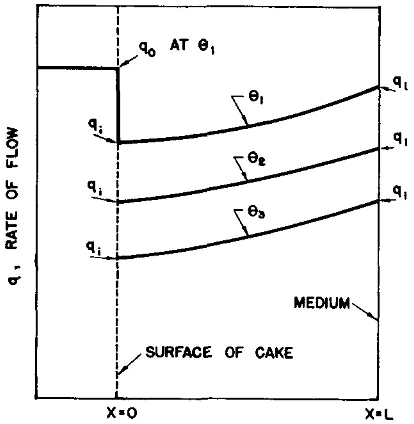

Fig. 9. Variation of flow rate with distance.

For convenience of calculation Equation (33) may be rewritten as

$$\epsilon_{av} = 1 - \frac{\displaystyle\int_o^{p-p_1} \frac{dp_s}{\alpha_x}}{\displaystyle\int_o^{p-p_1} \frac{dp_s}{\alpha_x(1-\epsilon)}} \tag{34}$$

Use of Equation (34) instead of Equation (33) permits a simplification in that the integral of $\epsilon\, dp_s/\alpha_x(1-\epsilon)$ does not have to be obtained. The value of m may be found by substituting ϵ_{av} from Equation (33) into Equation (5).

Substitution of ϵ_{av} in Equation (5), placing the resulting value of m in (28), and rearranging terms yields

$$vq = \frac{g_c}{\mu\rho s}[1 - s(1-\rho/\rho_s)] \int_o^{p-p_1} \frac{dp_s}{\alpha_x} - \frac{g_c}{\mu\rho_s} \int_o^{p-p_1} \frac{dp_s}{\alpha_x(1-\epsilon)} \tag{35}$$

which may be viewed as an improvement on Equation (8) presented in paper number III of this series (8). The last term in Equation (35) represents a correction factor for the variation of the average liquid content of the cake. When the cake porosity approaches a limiting average value, the last term combines with the first to yield the conventional filtration equation.

Calculations with the exit rate and based on (35) are illustrated in Figure 10 in the form of curves for talc (1) at 5.0 and 15 lb./sq. in. and polystyrene latex (1) at 10 lb./sq. in. The curves are plotted in the form of $d\theta/dv$ vs. v, and according to conventional theory such curves should yield straight lines if α, m, and R_m were constant. Variations in filtration resistance and average porosity during the initial period cause a marked deviation from straight-line plots. If the linear portions of the curves are extrapolated to the vertical axis, incorrect medium resistances will result. In the case of latex it would appear that the resistance was zero from an extrapolation. In evaluating medium resistances from experimental data great care should be employed in extrapolating $d\theta/dv$ vs. v data to zero volume.

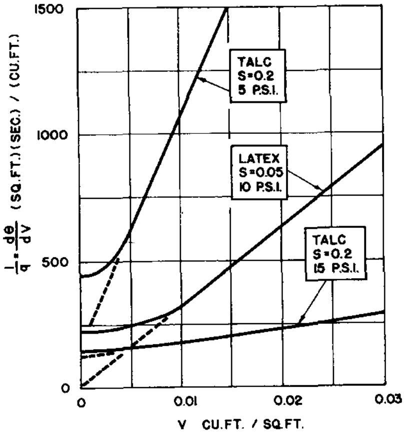

Fig. 10. Reciprocal rate vs. volume.

The additional work involved in the use of Equation (35) instead of more simple relations is only of value for the first few seconds of filtration. If the total time is measured in seconds as in rotary filtration, Equation (35) should be employed until simplifications have been justified. For longer filtration (35) reduces to the conventional Ruth equation in the form

$$vq_1 = \frac{(1-ms)\ g_c\ (p-p_1)}{\mu\ \rho\ s\ \alpha} \tag{36}$$

where q is replaced by the exit rate of flow q_1. If the other limiting value of the rate q_i is substituted in Equation (28), there results

$$vq_i = \frac{(1-ms)\,g_c\,(p-p_1)}{\mu\,\rho\,s\,\alpha} \tag{37}$$

Replacing p_1 by Equation (2), q_i by Equation (13), and rearranging one gets

$$\frac{\mu\rho s\alpha}{g_c p}\frac{(1-m_i s)}{(1-ms)^2}v\,dv + \frac{\mu R_m}{g_c p}dv = d\theta \tag{38}$$

If $m_i = m$ or $(1 - ms)$ is approximately unity, (38) reduces to the conventional equation. Integration of (38) produces

$$\frac{\mu\rho s\alpha}{g_c p}\frac{(1-m_i s)}{(1-ms)^2}\frac{v^2}{2} + \frac{\mu R_m}{g_c p}v = \theta_i \tag{39}$$

where θ_i represents the time when q_i is used. The difference in time between the conventional value and that given by (39) for the same volume is

$$\theta - \theta_i = \frac{\mu\rho s\alpha}{g_c p}\frac{1}{1-ms} - \frac{(1-m_i s)}{(1-ms)^2}\frac{v^2}{2} \tag{40}$$

If the medium resistance is negligible, the fractional difference becomes

$$\frac{\theta-\theta_i}{\theta} = 1 - \frac{1-m_i s}{1-ms} = \frac{(m_i - m)s}{1-ms} \tag{41}$$

The ratio given in (41) is exactly equal to $(q_1 - q_i)/q_1$.

Example

Talc (1) is to be filtered at a constant pressure of 15 lb. force/sq. in. under the following conditions:

μ = 0.001, lb. mass/(ft.)(sec.)
ρ = 62.4, lb. mass/cu. ft.
ρ_s = 167.0, lb. mass/cu. ft.
s = 0.20, mass fraction solids in slurry
α = 1.85 (10^{11}), ft./lb. mass (numerical integration)
ϵ_i = 0.900, extrapolation from 1.1 lb./sq. in.
ϵ_{av} = 0.826, numerical integration obtained with Equation (34)
m = 2.79, Equation (5)
m_i = 4.36, Equation (5) with ϵ_i substituted for ϵ_{av}

Neglect medium resistance and calculate the possible variation in time necessary to produce a cake 2 in. in thickness.

The mass of dry solids per square foot for a 2 in. cake is given by

$$w = \rho_s(1-\epsilon_{av})L = 4.84 \text{ lb./sq.ft.} \tag{42}$$

and the filtrate volume becomes

$$v = \frac{1-ms}{\rho s}w = 0.171 \text{ cu. ft./sq. ft.} \tag{43}$$

Calculating the conventional time one gets

$$\theta = \frac{\mu\,\rho\,s\,\alpha}{g_c p(1-ms)}\frac{v^2}{2} = 1{,}100 \text{ sec.} \tag{44}$$

To find the possible variation $(\theta - \theta_i)/\theta$ is calculated from (41) and found to be 0.712. Therefore the time based upon q_i as the rate throughout the cake is 782 sec. Determination of a more accurate value depends upon solution of the partial differential equation given in (26).

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NOTATION

f_i = void ratio at cake surface, dimensionless
f_{av} = average void ratio of cake, dimensionless
g_c = conversion factor, poundal/pound force, (lb.mass)(ft.)/(lb.force)(sec.^{2})
L = cake thickness, ft.
m = ratio of mass of wet to mass of dry cake, dimensionless
m_i = value of m in infinitesimal surface layer of cake
p = applied filtration pressure, lb. force/sq. ft.*
p_s = solid compressive pressure at distance x from surface of cake, also total compressive pressure, lb. force/sq. ft.
p_x = hydraulic pressure at distance x from surface of cake, lb. force/sq. ft.
p_1 = pressure at interface of medium and cake, lb. force/sq. ft.
q = rate of flow of liquid in cake at distance x from surface, cu. ft./(sq. ft.)(sec.)
q_1 = value of q at interface of medium and cake
q_i = value of q in infinitesimal surface layer of cake
q_0 = rate of flow of liquid in slurry prior to entry of cake surface, cu. ft./(sq. ft.)(sec.)
R_m = medium resistance, 1/ft.
s = fraction solids in slurry, dimensionless
v = volume of filtrate, cu. ft./sq. ft.
v_0 = volume of liquid entering cake from slurry, cu. ft./sq. ft.
v_i = volume of liquid leaving surface of cake, cu. ft./sq. ft.
x = distance from surface of cake, ft.
w = total mass of dry solids per unit area, lb. mass/sq. ft.
w_x = mass of solids per unit area in distance x from surface of cake, lb. mass/sq. ft.
X = pounds of solid per pound of slurry in cake, dimensionless

Greek Letters

α = average specific resistance, ft./lb. mass
α_x = value of specific resistance at distance x from cake surface where solid compressive pressure is p_s, lb. force/sq. ft.
ϵ = porosity at distance x from surface
ϵ_{av} = average porosity of cake
θ = time, sec.
θ_i = time defined by Equation (39), sec.
μ = viscosity, lb.mass/(ft.)(sec.)
ρ = density of liquid, lb. mass/cu. ft.
ρ_{av} = average cake density, $\rho\epsilon_{av} + \rho_s(1-\epsilon_{av})$, lb. mass/cu. ft.
ρ_s = true density of solids, lb. mass/cu. ft.
σ = ratio of ρ_s to ρ, dimensionless

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* Previously the author (*6, 7*) used capital *P* for pressure in lb. force/sq. ft. and lower case *p* for pressure expressed as lb. force/sq. in. A change was made in this paper because of the adoption of new standards for nomenclature by the American Institute of Chemical Engineers. See Mott Souders, *Chem. Eng. Progr.* **52**, 255 (1956).